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# Derivation of Tsallis statistics from dynamical equations for a granular gas

**F Sattin**

Consorzio RFX, Associazione Euratom-ENEA per la Fusione, Corso Stati Uniti 4,  
35127 Padova, Italy

E-mail: [sattin@igi.pd.cnr.it](mailto:sattin@igi.pd.cnr.it)

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## Abstract

In this paper we present the explicit calculation of the velocity probability distribution function  $P(v)$  for a model system of granular gas within the framework of Tsallis non-extensive statistical mechanics, using the stochastic approach by Beck (2001 *Phys. Rev. Lett.* **87** 180601), further generalized by Sattin and Salasnich (2002 *Phys. Rev. E* **65** 035106(R)). The calculation is self-consistent in that the form of  $P(v)$  is not given as an ansatz but is shown to necessarily arise from the known microscopic dynamics of the system.

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## 1. Introduction

Much research has recently been devoted to the study of spatially uniform systems with dissipative interactions ('granular matter'); see, for example, [1–6] and references therein. Among the issues addressed, much attention is given to transport properties and therefore to the details of the asymptotic velocity statistics. Although details vary according to the particular model chosen for describing the granular matter, an interesting feature common to almost all of these models is the overpopulation of the high-velocity tail of the probability distribution function (PDF) with respect to the standard Maxwellian (Gaussian) distribution, either in the form of a stretched exponential, or a power-law distribution [5].

The existence of non-Gaussian high-energy tails calls obviously for points of contact with the formalism of the non-extensive statistical mechanics (NESM), originally introduced by Tsallis [7], which has exactly this distinctive feature. Furthermore, there is the interesting property of some models for granular materials, so-called inelastic Maxwell models (IMM), allowing analytical calculations to be performed. The fact that velocity PDFs can be computed exactly from the constitutive equations of the model raises the interesting question: how is Tsallis statistics actually generated by the microscopic dynamics of a system? This point, in connection with granular gases, was first raised but left unanswered in [1].

In this paper, we address this question and show that PDFs for the class of granular gases studied can be computed self-consistently by essentially combining the microscopic dynamics with the hypothesis of the molecular chaos and with a stochastic formulation of NESM. To the best of our knowledge, only one other group [8] has been able up to now to explicitly derive Tsallis statistics from dynamical equations of an Hamiltonian system (but see also appendix A). However, there, the system was built *ad hoc*. In our case, instead, we are studying a model originally devised for a completely different purpose and which aims to be an approximate but realistic approximation of physical systems found in reality.

The structure of this paper is as follows. In section 2, we give a brief description of the IMM. Section 3 is devoted to the description of the stochastic interpretation of NESM. Calculations are performed in section 4. Finally, section 5 is devoted to conclusions.

## 2. The inelastic Maxwell model

The model we use, or some variant of it, has been extensively used for the study of one- or two-component granular materials [1–6]. Granular materials are assemblies of macroscopic particles interacting through inelastic collisions. They possess some uncommon features. Notably, they fail to obey the zeroth law of thermodynamics; in a mixture of two or more interacting components isolated from the outside, each component can reach a temperature (measured as the average kinetic energy of its elements) different from the other components, although in the presence of an exchange of energy among them [1, 2].

Let us consider an ensemble of  $N$  particles,  $N_1$  of species 1 and  $N_2$  of species 2. Let  $p = N_1/(N_1 + N_2) = N_1/N$ . The masses of the two species are different,  $m_1$  and  $m_2$  respectively, and we define  $\zeta = m_1/m_2$ . To fix ideas, we also let  $m_1 \geq m_2$  ( $\zeta \geq 1$ ). The particles collide between them at a rate independent of their velocity and only binary impulsive collisions are considered. Collisions can be inelastic and the degree of loss of energy in a collision between two particles of species  $\alpha$  and  $\beta$  is quantified by the restitution coefficient  $r_{\alpha\beta} \leq 1$ , with the equal sign holding for elastic collisions.

In one-dimensional geometry, the velocities of two particles  $i$  and  $j$  change after each collision according to

$$\begin{aligned} v_i^{(\alpha)} &= v_i^{(\alpha)} - (1 + r_{\alpha\beta}) \frac{m_\beta}{m_\alpha + m_\beta} (v_i^{(\alpha)} - v_j^{(\beta)}) \\ v_j^{(\beta)} &= v_j^{(\beta)} + (1 + r_{\alpha\beta}) \frac{m_\alpha}{m_\alpha + m_\beta} (v_i^{(\alpha)} - v_j^{(\beta)}). \end{aligned} \quad (1)$$

The superscript is to remind that the collision can be between like ( $\alpha = \beta$ ) or unlike ( $\alpha \neq \beta$ ) particles. It is possible to show that equations (1) conserve the momentum but not the energy, unless all  $r_{\alpha\beta}$  are equal to 1. All the particles are allowed to interact, regardless of their position (mean-field approximation). In numerical calculations, a predefined fraction of particles at each time step is randomly picked up and made to collide.

Let us now briefly describe the main features of this model. Master equations can be derived for velocity PDFs  $P_\alpha(v, t)$ , which give the probability density for finding a particle of species  $\alpha$  at time  $t$  with a velocity of around  $v$ . They satisfy a coupled set of differential equations; see [1, 4].

In the case of a pure component, i.e. only particles of species 1 or 2, it is possible to show [1] that an asymptotic solution (i.e., valid for  $t \rightarrow \infty$ ) of the corresponding equation for  $P(v, t)$  exists and has the shape

$$P_s(v, t) = \frac{2}{\pi} \frac{1}{v_0(t)} \frac{1}{\left[1 + \left(\frac{v}{v_0(t)}\right)^2\right]^2} \quad (2)$$

where  $v_0(t)$  is an exponentially decaying function,  $v_0(t) = v_0(0) \exp(-\lambda(r)t)$ , and  $\lambda$  is a function of the restitution coefficient with  $\lambda = 0$  if  $r = 1$ .

The result (2) is intriguing since it predicts a power-law tail for high  $v$ .  $P_s$  has the form commonly found within the framework of NESM but it derives from ‘exact’ equations of motion, and not from variational principles involving the entropy and the energy of the system.

The two-component case is more complicated and equations can be solved analytically only to a certain extent and under some approximations. The  $p \rightarrow 0$  limit was addressed by Ben-Naim and Krapivsky [4]. This limit allows one population (the background fluid) to be freely evolving while the other (the impurity) is driven by the former. For  $\zeta \gg 1$  (i.e., light impurities embedded into a fluid of heavy particles), Ben-Naim and Krapivsky demonstrated that the PDFs for the impurities as well as for the fluid can generically be written as linear combinations of powers of Lorentzian functions:

$$P_{\text{fluid,imp}}(v) = \sum_{n \geq 2} B_{\text{fluid,imp}}^{(n)} \left( \frac{1}{1 + v^2} \right)^n \tag{3}$$

The coefficients  $B^{(n)}$  depend upon the restitution coefficients, and some or even infinitely many of them (but, of course, not all) may be zero. In the limit of high  $v$ , only the leading term survives, hence  $P \sim 1/v^4$  but for particular combinations of the parameters. Of course, the result (3) encompasses equation (2) since  $p = 0$  corresponds to the pure case.

It is worthwhile mentioning that Bettolo Marconi and Puglisi [1] also addressed the two-component problem, still assuming  $\zeta \gg 1$ , but allowing for finite values of  $p$ . They found numerically that both species of particles have leading terms for high- $v$  tails of the form

$$P_{\alpha,\beta}(v, t) \approx \frac{1}{\left[ 1 + \left( \frac{v}{v_0(t)} \right)^2 \right]^{\varepsilon(\alpha,\beta)}} \tag{4}$$

with exponents  $\varepsilon$  which depend on all the parameters  $p, r_{\alpha\beta}, \zeta$  in a complicated fashion. However, while  $\varepsilon(1) = 2$ , the light component was found to have a steeper descent:  $\varepsilon(2) \sim 3$ . We argue that the inconsistency with the results [4] is only apparent, and the result (4) is a particular case of equation (3), corresponding to a peculiar choice of parameters such that  $B^{(2)} \sim 0$ . The calculation will be put forth in appendix B.

Before ending this section, we elaborate on equations (1) that will be useful for the following. By defining  $v_i^{(\alpha)} - v_j^{(\beta)} = v_{\text{rel}}, (1 + r_{\alpha\beta}) \frac{m_\beta}{m_\alpha + m_\beta} = f_{\alpha\beta}$ , we reorder the former of equation (1):

$$\Delta v_i^{(\alpha)} = -f_{\alpha\beta} v_{\text{rel}} \tag{5}$$

From now on we consider formally the right-hand side of equation (5) to be a known function. By averaging over several collisions it is clear that equation (5) describes a Brownian motion in the velocity space, of step  $|f_{\alpha\beta} v_{\text{rel}}|$ . It is convenient to turn equation (5) into the form of a differential equation

$$\frac{dv^{(\alpha)}}{dt} = |f_{\alpha\beta} v_{\text{rel}}| L(t) \tag{6}$$

with  $L(t)$  white noise of zero mean and whose variance may be chosen to be unity after a suitable choice of time scale. The minus sign appearing in equation (5) has been absorbed into the definition of  $L$ .

At this point it is convenient to formally turn the system (6) into a driven one by adding a term  $F = -\gamma v^{(\alpha)}$ . This turns equation (6) into the Langevin equation

$$\frac{dv^{(\alpha)}}{dt} = -\gamma v^{(\alpha)} + |f_{\alpha\beta} v_{\text{rel}}| L(t) \tag{7}$$

from which the asymptotic temperature can be computed:

$$\frac{1}{\beta} = \frac{f_{\alpha\beta}^2 v_{\text{rel}}^2}{\gamma}. \quad (8)$$

We anticipate that  $\beta$  will turn into a fluctuating quantity, because of its dependency on the stochastic variable  $v_{\text{rel}}$ .

A non-zero value of  $\gamma$  corresponds to providing energy to (or draining from) the system by the outside. Since in a dissipative system  $v_{\text{rel}}$  approaches zero asymptotically, we can let  $\gamma \rightarrow 0$  while simultaneously maintaining finite the ratio (8). In this way,  $\beta$  is defined but for a multiplicative constant, which is of no relevance to our purposes. Notice, however, that the pure elastic case ( $r_{\alpha\beta} = 1$ ) cannot, in this way, be included in our model, since it would correspond to  $\gamma$  rigorously zero, thus equation (8) can no longer hold. The elastic limit is a singular limit here because we are dealing with asymptotic behaviour; regardless of the smallness of dissipation, it will always lead to finite effects over infinitely long times.

### 3. The stochastic interpretation of NESM

Tsallis NESM has received considerable interest over the years because of its capability to describe a wealth of disparate phenomena (from anomalous diffusion, to turbulent systems, etc) apparently unexplainable within the framework of the standard statistical mechanics. A constantly updated list of references can be found in [9]; some recent reviews on the subject are also given [10].

A subject relatively less explored concerns the relationship between NESM and the microscopic dynamical properties of a given system; we would like to be able to compute the PDF for one system given its evolution equations, or, at least, uniquely relate the features of the PDF (say, the non-extensive exponent  $q$ ) to some microscopical property characterizing the system. We mention here the paper by Wilk and Włodarczyk [11], which first related the non-extensive exponent  $q$  to a definite microscopic property; in their case, the fluctuations of the cross section for stopping high-energy cosmic rays. Beck [12] provided later a theoretical framework to their finding, and Almeida [13] reached a similar conclusion by a different path. More recently, Sattin and Salasnich [14] generalized Beck's approach. Finally, Beck and Cohen [15] formalized the whole by introducing the concept of superstatistics.

We mention also that a deterministic connection between NESM and microdynamics was provided in a recent work (Adib *et al* [8]), showing how power-law PDFs arise in mesoscopic systems evolving under suitable homogeneous Hamiltonians; an explicit calculation of a power-law PDF was performed using a Hamiltonian of the Fermi–Pasta–Ulam type.

It is useful to briefly summarize the results [12, 14, 15], which are used in this paper. According to Beck, any system  $S$  endowed with Hamiltonian  $H$  in equilibrium with a surrounding heat reservoir  $B$  follows the standard Boltzmann statistics, thus its canonical distribution is given by the Maxwell–Gibbs exponent:  $\rho(H) \propto \exp(-\beta H)$ . However, the reservoir itself may be not a simple system in equilibrium, but instead consisting of several subsystems interacting between them. The generalized temperature  $1/\beta$ , which quantifies the interaction between  $S$  and  $B$ , might therefore not be a constant but a function varying in time, to all extents a stochastic variable characterized by its own PDF  $P(\beta)$ . Therefore, any attempt to determine  $\rho(H)$  would yield only its average over  $P(\beta)$ :

$$\rho(H) = \int d\beta C_H(\beta) \exp(-\beta H) P(\beta). \quad (9)$$

$C_H(\beta)$  is the normalization constant for  $\rho(H)$ , which depends in general both on  $\beta$  as well as on the particular form for  $H$ . In his original paper [12], Beck argues that  $P(\beta)$  should belong

to a particular class of functions ( $\chi^2$ -distributions). It is straightforward to show that, using the standard free-particle Hamiltonian  $H = u^2/2$ , we recover the power-law behaviour for  $\rho(u)$  first predicted by Tsallis on the basis of the maximum entropy principle. If, instead,  $P(\beta)$  is a Dirac delta,  $P(\beta) = \delta(\beta - \beta_0)$ , i.e. when fluctuations of  $\beta$  around its average value are small, the standard extensive statistics is recovered. The two limits are not exclusive since a  $\chi^2$ -function with infinite degrees of freedom approaches a Dirac delta.

Beck's is an ansatz about  $P(\beta)$ . In [14, 15] it is shown that if, instead, this constraint is relaxed and  $\beta$  is supposed to be a derived quantity,  $\beta = \beta(c_1, c_2, \dots)$ , where  $c_i$  are the actual fluctuating parameters that govern the dynamics of the systems, it is possible to widen the class of distributions  $P(\beta)$  may belong to. As a consequence, different, even non-power-law functional forms for  $\rho(u)$ , were obtained. It was shown in [14] that some experimental distributions could be fitted using these generalized functional forms.

#### 4. Calculation of velocity statistics

Our purpose in this paper is to recover, within the interpretation for NESM described in section 3, the results of [4], namely that: (i) the PDF for a pure granular gas has a power-like high- $v$  tail; (ii) the PDF for a light impurity in a background fluid has the same shape as that of the fluid. We point out from the outset that our treatment will fall short of: (i) delivering the value for the exponent of the power law; (ii) giving an account of the particular cases found in [1] where the impurity and the fluid have different exponents. We believe, and try to demonstrate in appendix B, that this latter shortcoming is a minor one since these cases should be rather exceptional, i.e., of null measure over the parameter space.

We will begin with the pure case. The starting point is equation (9), hence we must know  $P(\beta)$  in order to estimate the velocity PDF  $P(v)$ . At this stage, there are two possible ways. One way is to accept original the Beck ansatz and postulate that  $P(\beta)$  is a  $\chi^2$ -distribution

$$P(\beta) = \frac{1}{\Gamma\left(\frac{n}{2}\right)} \left[\frac{n}{2\beta_0}\right]^{n/2} \beta^{n/2-1} \exp\left(-\frac{n\beta}{2\beta_0}\right) \quad (10)$$

where  $n$  is the number of degrees of freedom and  $\beta_0$  is the average value of  $\beta$ .

If we replace expression (10) in (9), and for  $H$  take the free-particle Hamiltonian,  $H = v^2/2$ , we can write

$$P(v) = K \int \beta^{1/2} \exp(-\beta v^2/2) P(\beta) d\beta \quad (11)$$

where the term  $\beta^{1/2}$  stems from the normalization of  $\rho(H)$ , and all other constant terms have been collected into  $K$ . It is straightforward to see that the result of the integration is

$$P(v) = K \frac{1}{\left[1 + \left(\frac{v}{v_0}\right)^2\right]^{\frac{n+1}{2}}} \quad (12)$$

and hence the result (3) is obtained for  $(n+1)/2 = 2 \rightarrow n = 3$ .

This path is not completely satisfactory since it leaves us with at least two unanswered questions: (i) why does  $P(\beta)$  have the form (10) and (ii) why does  $n$  take exactly that value?

The other approach is patterned following [14, 15]. Rather than postulating a form for  $P(\beta)$ , we attempt to derive one from what we know of the system. This way, we will be able to give an answer to question (i), although not to (ii).

In equation (8) a relationship was written between  $\beta$  and the relative impact velocity. Now we recall the relation between  $v_{\text{rel}}$  and the particle velocity and we point out that the latter is actually a stochastic variable with its own PDF. Therefore,  $v_{\text{rel}}$  and  $\beta$  must also be considered

as stochastic variables.  $\beta$ , in particular, inherits its PDF from that of  $v_{\text{rel}}$ ; if  $P(v_{\text{rel}})$  and  $P(\beta)$  are the PDFs for  $v_{\text{rel}}$  and  $\beta$ , a simple relationship exists:

$$P(\beta) d\beta = P(v_{\text{rel}}(\beta)) \left| \frac{dv_{\text{rel}}}{d\beta} \right| d\beta \sim P(v_{\text{rel}}(\beta)) \frac{1}{\beta^{3/2}} d\beta. \quad (13)$$

Since  $P(v_{\text{rel}})$  is related to  $P(v)$ ,  $P(\beta)$  is some, still unknown, function of  $P(v)$ .

Equation (11) with  $P(\beta)$  given by equation (13) is actually an implicit equation for  $P(v)$ . To solve it, we guess an initial functional form for  $P(v)$ , we use it for computing  $P(v_{\text{rel}})$ ,  $P(\beta)$  and hence, using equation (11), we recover  $P(v)$  again; the correct PDF must be a fixed point of this sequence of transformations.

As a starting guess, obviously we adopt the expression (4). The joint probability for two particles  $i$  and  $j$  (assumed uncorrelated) to have the velocities  $v_i$ ,  $v_j$  is therefore

$$P(v_i)P(v_j) dv_i dv_j = K \frac{1}{\left[1 + \left(\frac{v_i}{v_i^{(0)}}\right)^2\right]^\sigma} \frac{1}{\left[1 + \left(\frac{v_j}{v_j^{(0)}}\right)^2\right]^\sigma} dv_i dv_j. \quad (14)$$

Since the two particles are of the same species, they have the same PDF, and hence we have chosen the same exponent  $\sigma$  for both. Also, the average velocity must be the same, thus  $v_i^{(0)} = v_j^{(0)} \equiv 1$  after suitable rescaling. By converting to the variables  $v_{\text{rel}} = v_i - v_j$ ,  $y = v_j$  we can obtain the marginal probability  $P(v_{\text{rel}})$ :

$$P(v_{\text{rel}}) dv_{\text{rel}} = K \left( \int \frac{1}{[1 + (v_{\text{rel}} + y)^2]^\sigma} \frac{1}{[1 + y^2]^\sigma} dy \right) dv_{\text{rel}}. \quad (15)$$

From here on, all irrelevant constant terms must be considered as automatically merged into the constant  $K$ , without the need to explicitly redefine it each time. The integral within parentheses can be analytically performed. However, let us recall that we are interested in the high-energy tail of  $P(v)$ ; therefore only the asymptotic behaviour for  $\beta \rightarrow 0$  and thus for  $v_{\text{rel}} \rightarrow \infty$  is of interest, and it can easily be shown that

$$P(v_{\text{rel}}) dv_{\text{rel}} \xrightarrow{v_{\text{rel}} \rightarrow \infty} \frac{K}{v_{\text{rel}}^{2\sigma}} dv_{\text{rel}}. \quad (16)$$

From equations (8) and (13) we obtain

$$\lim_{\beta \rightarrow 0} P(\beta) = K\beta^{\sigma-3/2} \quad (17)$$

and, substituting into equation (11),

$$P(v) = K \int \beta^{\sigma-1} \exp(-\beta v^2/2) d\beta \sim \frac{1}{(v^2)^\sigma}. \quad (18)$$

We thus return to the initial expression (14). This is not a rigorous demonstration, since we have not shown that there is only one solution. However, the several constraints that must be satisfied by any physically realizable solution (it must be continuous with at least some of its derivatives, normalizable, monotonically decreasing for high  $v$ ) severely restrict the class of potential candidates. It seems, therefore, enough to show that some other apparently plausible choices fail instead to satisfy the above equations. We thus repeat calculations for the most obvious choice: the Gaussian PDF.

We can straightforwardly write the equivalent of equation (15)

$$\begin{aligned} P(v_{\text{rel}}) dv_{\text{rel}} &= K \left[ \int \exp(-(v_{\text{rel}} + y)^2 - y^2) dy \right] dv_{\text{rel}} \\ &= K \exp(-v_{\text{rel}}^2/2) dv_{\text{rel}} \\ \rightarrow P(\beta) &= K \frac{1}{\beta^{3/2}} \exp\left(-\frac{\beta_0}{\beta}\right) \end{aligned} \quad (19)$$

with  $\beta_0$  constant term. Hence,

$$P(v) = \int \beta^{-1} \exp(-\beta v^2/2 - \beta_0/\beta) d\beta \approx K_0(\sqrt{2\beta_0 v^2}) \quad (20)$$

$$\sim \frac{1}{v} e^{-\sqrt{2\beta_0}v} \quad (v \rightarrow \infty)$$

with  $K_0$  modified Bessel function of order 0. Since we do not recover the initial PDF, this choice must be ruled out. Another PDF frequently used in this field is the stretched exponential  $P(v) \propto \exp(-c|v|^\alpha)$ , with  $\alpha < 2$ . The general case cannot be dealt with, since it is not possible to analytically integrate equation (19). However, the case  $\alpha = 1$  is easy to compute, and we recover the same result.

We do not deal with the two-component case with substantial differences. We limit ourselves to vanishing fractions of impurities,  $p \sim 0$ . This means that, in equation (1), an impurity particle is likely to collide only with a fluid particle. The relative velocity becomes therefore  $v_{\text{rel}} = v_{\text{imp}} - v_{\text{fluid}}$ . We allow for the two populations to have similar PDFs but with different exponents. Thus equation (14) becomes

$$P(v_{\text{imp}})P(v_{\text{fluid}}) dv_{\text{imp}} dv_{\text{fluid}} = K \frac{1}{\left[1 + \left(\frac{v_{\text{imp}}}{v_{\text{imp}}^{(0)}}\right)^2\right]^{\sigma_{\text{imp}}}} \frac{1}{\left[1 + \left(\frac{v_{\text{fluid}}}{v_{\text{fluid}}^{(0)}}\right)^2\right]^{\sigma_{\text{fluid}}}} dv_{\text{imp}} dv_{\text{fluid}}. \quad (21)$$

Now the two average velocities  $v_{\text{imp,fluid}}^{(0)}$  may in principle be different. Results [1] suggest that they actually are equal, but this is not of concern here. Calculation of the marginal probability  $P(v_{\text{rel}})$  shows that its asymptotic behaviour is, like equation (16),

$$P(v_{\text{rel}}) \sim \frac{1}{(v_{\text{rel}}^2)^{\sigma_{\text{fluid}}}} \quad (22)$$

irrespective of the value  $\sigma_{\text{imp}}$ . The rest of the calculation then continues just as in equations (17) and (18), showing that the high-energy tail is completely determined by the background fluid.

## 5. Conclusions

In this paper we have been able to provide a self-consistent derivation of NESM PDFs from microscopic dynamics, thereby providing a confirmation that Tsallis NESM is more than a useful but mere formalism. Rather, the appearance of non-standard statistics is a natural consequence of basic principles.

We point out that, even though the straightforward application of the naive Beck approach [12] would be sufficient to recover the sought result, much more physical insight is gained by the approach of Sattin and Salasnich.

Our results are not yet fully satisfactory. The most serious shortcoming, in our opinion, is that NESM formalism looks unable to provide self-consistently a quantitative estimate for its characteristic parameters (the exponent  $\sigma$  in our case,  $q$  in the general case). We cannot say if this is due to some deficiency of our own approach, which is not able to fully exploit all the information available, or it is inherent to NESM framework, although recent work [16] strongly suggests that  $q$  is defined by the environment.

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## Appendix A

It is useful to comment on the connections between this work and the Fokker–Planck equation approach to Tsallis statistics done by Borland.

In the paper [17], Borland shows that Tsallis-type macroscopic distributions  $P(x)$  can be obtained as stationary solutions of Fokker–Planck equations (or of equivalent Langevin equations)

$$\frac{d}{dt}P(x, t) = \frac{d}{dx}(K(x), P(x, t)) + \frac{1}{2} \frac{d^2}{dx^2}(D(x)P(x, t)) \quad (\text{A1})$$

provided that the convection and diffusion coefficients,  $K$  and  $D$ , are suitable functions of the independent variable  $x$ . Borland gives general conditions that must be fulfilled by the couple of functions  $(K(x), D(x))$ , as well as showing several particular cases. This approach and that of Beck (equation (10)), and therefore the approach of Sattin and Salasnich, clearly have many similarities. However, they are not equivalent; in particular, Borland deals with *one* equation, whose coefficients are functions of the independent variable. Beck's approach is instead that of considering an *ensemble* of equations, each of them with constant coefficients.

## Appendix B

To start with, we summarize the set of parameters used in the simulations [1]. These are: (i)  $\zeta \gg 1$  (very light impurities); (ii)  $p = 1/2$  (equal fractions of impurities and fluid); (iii)  $r_{\alpha\beta} \equiv r \sim 1 \forall \alpha, \beta$  (quasi-elastic restitution coefficients, independent of the species involved in the collision; the precise value reported in [1] was 0.95).

Conditions (i) and (iii) allow us to define the coefficients (equations (3) and (4) of [4])

$$P = \frac{1-r}{2} \sim 0 \quad Q = \frac{\frac{1}{\xi} - r}{1 + \frac{1}{\xi}} \sim -r \sim -1. \quad (\text{B1})$$

We give now without proof some results from [4]: (i) the coefficients  $B^{(n)}$  appearing in equation (3) can be written as linear combinations of other coefficients  $A_n$ . Here we are interested only in  $B^{(2)}$ ; its form is shown to be

$$B^{(2)} = 1 - 3A_2 + 3A_3. \quad (\text{B2})$$

The coefficients  $A_n$  satisfy a recursion relation

$$A_n = \frac{Q^{n-1}(1-Q) - P(1-P)}{1 - Q^n - nP(1-P)} A_{n-1} \quad (\text{B3})$$

where  $P$  and  $Q$  are defined in equation (B1), and which is supplemented by the initial condition  $A_0 = 1$ . It is straightforward to work out for our case

$$A_2 \sim \frac{-2}{1-Q^2} \rightarrow |A_2| \gg 1 \quad \text{and} \quad A_3 \sim A_2 \quad (\text{B4})$$

which, when replaced in equation (B2), give  $B^{(2)} \sim 0$ .

The set of parameters chosen in [1] is therefore such to almost cancel the coefficient of the leading term  $\sim 1/(1+v^2)^2$ . Although this term will eventually dominate for large enough  $v$ , numerical simulations are unable to sample effectively that region, and instead are dominated for finite  $v$  by the next-to-leading term  $1/(1+v^2)^3$ , whose coefficient is much larger.

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